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1 Direct Stiffness Method

1.1 Continuous beam

Use Direct Stiffness Method to calculate the internal forces of the continuous beam with rectangular cross-section 0.4×0.6 m and draw them. The beam is at the both end fixed (clamped, built-in, made-in, restrained) and in the middle of the span there is a mobile support. The loading of the beam consists of a single force (point load) 20 kN at angle 60 °, uniform load 5 kNm^{-1} in length 2 m and a single moment load 10 kNm (2 meters from the support b). See Fig. 1. The material of this beam has modulus of elasticity $E = 36\,\mathrm{GPa}$.

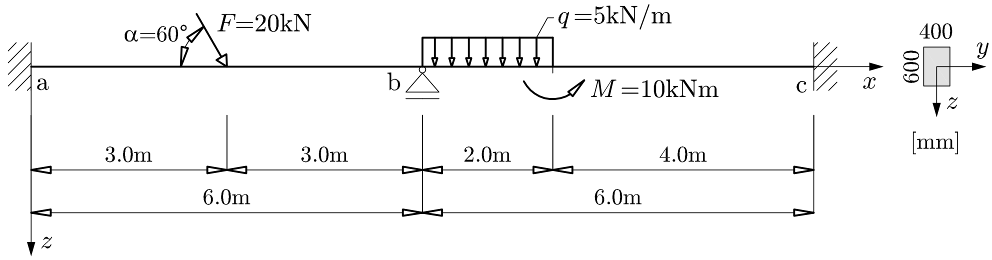

Fig. 1: Task scheme

We can write basic equation system:

$$[\boldsymbol{k}] \cdot \{\boldsymbol{r}\} = \{\boldsymbol{F}\} \tag{1}$$

where $[\boldsymbol{k}]$ = stiffness matrix of the bar system, $\{\boldsymbol{F}\}$ = force vector, $\{\boldsymbol{r}\}$ = displacement vector.
Cross-sectional area

$$A = b \cdot h = 0.4 \cdot 0.6 = 0.24\,\mathrm{m}^2$$

The second moment of area

$$I_{\mathrm{y}} = \frac{1}{12} \cdot b \cdot h^3 = \frac{1}{12} \cdot 0.4 \cdot 0.6^3 = 0.0072\,\mathrm{m}^4$$

1. We label nodes a, b, c and calculate degree of kinematic indeterminacy n_{k} (degrees of freedom). The kinematic degree of freedom is the number of independent joint displacements (rotations and translations). In a planar task every joint has three degrees of freedom (u_i, w_i, φ_i). One translation in the global X-direction, one translation in the Y-direction and one rotation about Z-axis (counter clockwise). Displacement vector $\{\boldsymbol{r}\}$ consists of unknown degrees of freedom, in our case $\{\boldsymbol{r}\} = \{u_{\mathrm{b}}, \varphi_{\mathrm{b}}\}^{\mathrm{T}} \Rightarrow n_{\mathrm{k}} = 2$. We know displacement of some nodes (e.g. $u_{\mathrm{a}} = 0$ node a is supported in X-direction, etc.). See Fig. 2.

$$\{\boldsymbol{r}\} = \left\{ \begin{array}{c} u_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \end{array} \right\} \tag{2}$$

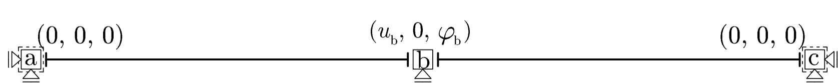

Fig. 2: Labels of nodes and degrees of freedom

2. We divide our continuous beam into two bars (members) a-b and b-c see Fig. 3.

$$F_{\mathrm{x}} = F \cdot \cos(\alpha) = 20 \cdot \cos(60\,°) = 10\,\mathrm{kN}$$
$$F_{\mathrm{z}} = F \cdot \sin(\alpha) = 20 \cdot \sin(60\,°) = 17.32\,\mathrm{kN}$$

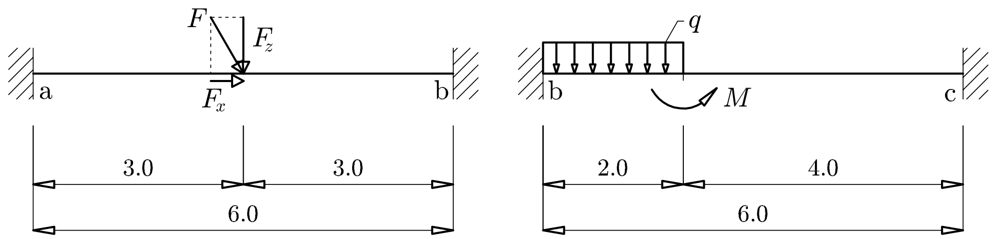

Fig. 3: Division of the beam into two bars a-b and b-c

To calculate stiffness matrix of members – we use tab. 11.3a, because bars a-b and b-c are fixed supported on both ends. The matrix is symmetric. The stiffness matrices $[\boldsymbol{k}_{\mathrm{ab}}]$ and $[\boldsymbol{k}_{\mathrm{bc}}]$ are identical by reason of the same bar length, cross-sectional area, the second moment of area and modulus of elasticity.

$$[\boldsymbol{k}_{\mathrm{ab}}] = [\boldsymbol{k}_{\mathrm{bc}}] =$$

$$= \begin{pmatrix} 1.44 & 0 & 0 & -1.44 & 0 & 0 \\ 0 & 0.0144 & -0.0432 & 0 & -0.0144 & -0.0432 \\ 0 & -0.0432 & 0.1728 & 0 & 0.432 & 0.0864 \\ -1.44 & 0 & 0 & 1.44 & 0 & 0 \\ 0 & -0.0144 & 0.0432 & 0 & 0.0144 & 0.0432 \\ 0 & -0.0432 & 0.0864 & 0 & 0.0432 & 0.1728 \end{pmatrix} \left[\cdot 10^9 \frac{\mathrm{N}}{\mathrm{m}}\right]$$

When we solve only displacement vector $\{\boldsymbol{r}\}$, we need only some matrix elements of $[\boldsymbol{k}]$ and $\{\bar{\boldsymbol{R}}\}$. These elements are highlighted below this text and in the next step 3.

$$[\boldsymbol{k}_{\mathrm{ab}}] =$$

$$\begin{array}{cc} \begin{array}{cccccc} u_{\mathrm{a}}(0) & w_{\mathrm{a}}(0) & \varphi_{\mathrm{a}}(0) & u_{\mathrm{b}}(1) & w_{\mathrm{b}}(0) & \varphi_{\mathrm{b}}(2) \end{array} & \\ \begin{pmatrix} 1.44 & 0 & 0 & -1.44 & 0 & 0 \\ 0 & 0.0144 & -0.0432 & 0 & -0.0144 & -0.0432 \\ 0 & -0.0432 & 0.1728 & 0 & 0.0432 & 0.0864 \\ -1.44 & 0 & 0 & 1.44 & 0 & 0 \\ 0 & -0.0144 & 0.0432 & 0 & 0.0144 & 0.0432 \\ 0 & -0.0432 & 0.0864 & 0 & 0.0432 & 0.1728 \end{pmatrix} & \begin{array}{c} u_{\mathrm{a}}(0) \\ w_{\mathrm{a}}(0) \\ \varphi_{\mathrm{a}}(0) \\ u_{\mathrm{b}}(1) \\ w_{\mathrm{b}}(0) \\ \varphi_{\mathrm{b}}(2) \end{array} \end{array} \left[\cdot 10^9 \tfrac{\mathrm{N}}{\mathrm{m}}\right]$$

$$[\boldsymbol{k}_{\mathrm{bc}}] =$$

$$\begin{array}{cc} \begin{array}{cccccc} u_{\mathrm{b}}(1) & w_{\mathrm{b}}(0) & \varphi_{\mathrm{b}}(2) & u_{\mathrm{c}}(0) & w_{\mathrm{c}}(0) & \varphi_{\mathrm{c}}(0) \end{array} & \\ \begin{pmatrix} 1.44 & 0 & 0 & -1.44 & 0 & 0 \\ 0 & 0.0144 & -0.0432 & 0 & -0.0144 & -0.0432 \\ 0 & -0.0432 & 0.1728 & 0 & 0.0432 & 0.0864 \\ -1.44 & 0 & 0 & 1.44 & 0 & 0 \\ 0 & -0.0144 & 0.0432 & 0 & 0.0144 & 0.0432 \\ 0 & -0.0432 & 0.0864 & 0 & 0.0432 & 0.1728 \end{pmatrix} & \begin{array}{c} u_{\mathrm{b}}(1) \\ w_{\mathrm{b}}(0) \\ \varphi_{\mathrm{b}}(2) \\ u_{\mathrm{c}}(0) \\ w_{\mathrm{c}}(0) \\ \varphi_{\mathrm{c}}(0) \end{array} \end{array} \left[\cdot 10^9 \tfrac{\mathrm{N}}{\mathrm{m}}\right]$$

Using highlighted matrix elements, we will assemble global stiffness matrix of the system

$$[\boldsymbol{k}] = \left(\begin{array}{c|c} 1.44 \cdot 10^9 & 0 \\ +1.44 \cdot 10^9 & +0 \\ \hline 0 & 1.728 \cdot 10^8 \\ +0 & +1.728 \cdot 10^8 \end{array}\right) = \begin{pmatrix} 2.88 \cdot 10^9 & 0 \\ 0 & 3.456 \cdot 10^8 \end{pmatrix} \qquad (3)$$

3. Force vector $\{\boldsymbol{F}\}$ is equal to nodal force vector $\{\boldsymbol{S}\}$ take away primary force vector $\{\bar{\boldsymbol{R}}\}$ (the strip above R signify that the vector is primary vector). In our case there is no nodal load this implies $\{\boldsymbol{S}\} = \emptyset$.

$$\{\boldsymbol{F}\} = \{\boldsymbol{S}\} - \{\bar{\boldsymbol{R}}\} \tag{4}$$

Primary vector $\{\bar{\boldsymbol{R}}_{\mathrm{ab}}\}$ of single force - see tab. 11.2c (14.10 row 2)

$$\{\bar{\boldsymbol{R}}_{ab}\} = \left\{\begin{array}{c} \bar{X}_{\mathrm{ab}} \\ \bar{Z}_{\mathrm{ab}} \\ \bar{M}_{\mathrm{ab}} \\ \bar{X}_{\mathrm{ba}} \\ \bar{Z}_{\mathrm{ba}} \\ \bar{M}_{\mathrm{ba}} \end{array}\right\} = \left\{\begin{array}{c} -5000 \\ -8660 \\ 12990 \\ -5000 \\ -8660 \\ -12990 \end{array}\right\} [\mathrm{N}] \tag{5}$$

Primary vector $\{\bar{\boldsymbol{R}}^1_{\mathrm{bc}}\}$ of uniform load in length 2 m - see tab. 14.10(row 11 – formulas are given in the second column)

$$\{\bar{\boldsymbol{R}}^1_{\mathrm{bc}}\} = \left\{\begin{array}{c} \bar{X}^1_{\mathrm{bc}} \\ \bar{Z}^1_{\mathrm{bc}} \\ \bar{M}^1_{\mathrm{bc}} \\ \bar{X}^1_{\mathrm{cb}} \\ \bar{Z}^1_{\mathrm{cb}} \\ \bar{M}^1_{\mathrm{cb}} \end{array}\right\} = \left\{\begin{array}{c|c} 0 & 0 \\ -9074 & -\frac{q\cdot a}{2\cdot l^3}\left[2\cdot l\cdot\left(l^2-a^2\right)+a^3\right] \\ 6111 & \frac{q\cdot a^2}{12\cdot l^2}\left(6\cdot b^2+3\cdot a\cdot b+a\cdot l\right) \\ 0 & 0 \\ -926 & -\frac{q\cdot a^3}{2\cdot l^3}\left(l+b\right) \\ -1667 & -\frac{q\cdot a^3}{12\cdot l^3}\left(3\cdot b-l\right) \end{array}\right\} [\mathrm{N}]$$

Primary vector $\{\bar{\boldsymbol{R}}^2_{\mathrm{bc}}\}$ of single moment load - see tab. 11.2d (14.10 row 9)

$$\{\bar{\boldsymbol{R}}^2_{bc}\} = \left\{\begin{array}{c} \bar{X}^2_{\mathrm{bc}} \\ \bar{Z}^2_{\mathrm{bc}} \\ \bar{M}^2_{\mathrm{bc}} \\ \bar{X}^2_{\mathrm{cb}} \\ \bar{Z}^2_{\mathrm{cb}} \\ \bar{M}^2_{\mathrm{cb}} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ -2222 \\ 0 \\ 0 \\ 2222 \\ 3333 \end{array}\right\} [\mathrm{N}]$$

Primary vector $\{\bar{\boldsymbol{R}}_{\mathrm{bc}}\}$

$$\{\bar{\boldsymbol{R}}_{\mathrm{bc}}\} = \{\bar{\boldsymbol{R}}^1_{\mathrm{bc}}\} + \{\bar{\boldsymbol{R}}^2_{\mathrm{bc}}\} = \left\{\begin{array}{c} \bar{X}_{\mathrm{bc}} \\ \bar{Z}_{\mathrm{bc}} \\ \bar{M}_{\mathrm{bc}} \\ \bar{X}_{\mathrm{cb}} \\ \bar{Z}_{\mathrm{cb}} \\ \bar{M}_{\mathrm{cb}} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ -11296 \\ 6111 \\ 0 \\ 1296 \\ 1666 \end{array}\right\} [\mathrm{N}] \tag{6}$$

Force vector $\{\boldsymbol{F}\}$ see eq. 4. We use only elements in relation to unknown displacements u_{b} and φ_{b}.

$$\{\mathbf{F}\} = \{\emptyset\} - \left\{\begin{array}{c} -5000+0 \\ -12990+6111 \end{array}\right\} = \left\{\begin{array}{c} 5000 \\ 6879 \end{array}\right\}$$

4. Solution of vector of unknown displacements $\{\boldsymbol{r}\}$

$$\begin{pmatrix} 2.88\cdot 10^9 & 0 \\ 0 & 3.456\cdot 10^8 \end{pmatrix} \cdot \left\{\begin{array}{c} u_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \end{array}\right\} = \left\{\begin{array}{c} 5000 \\ 6879 \end{array}\right\}$$

$$\left\{\begin{array}{c} u_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \end{array}\right\} = \left\{\begin{array}{c} 1.736\cdot 10^{-6} \\ 19.905\cdot 10^{-6} \end{array}\right\} \left[\begin{array}{c} \mathrm{m} \\ \mathrm{rad} \end{array}\right]$$

5. Solution of end reaction vector. $\left\{\widehat{\boldsymbol{R}}_{\mathrm{bc}}\right\}$ is the secondary force vector. To solve this equation system we need only highlighted columns, which are in relation to displacements differing from 0.

$$\{\boldsymbol{R}_{\mathrm{ab}}\} = \{\bar{\boldsymbol{R}}_{\mathrm{ab}}\} + \left\{\widehat{\boldsymbol{R}}_{\mathrm{ab}}\right\} = \{\bar{\boldsymbol{R}}_{\mathrm{ab}}\} + [\boldsymbol{k}_{\mathrm{ab}}] \cdot \{\boldsymbol{r}_{\mathrm{ab}}\} \tag{7}$$

$$\{\boldsymbol{R}_{\mathrm{bc}}\} = \{\bar{\boldsymbol{R}}_{\mathrm{bc}}\} + \left\{\widehat{\boldsymbol{R}}_{\mathrm{bc}}\right\} = \{\bar{\boldsymbol{R}}_{\mathrm{bc}}\} + [\boldsymbol{k}_{\mathrm{bc}}] \cdot \{\boldsymbol{r}_{\mathrm{bc}}\} \tag{8}$$

$$\{\boldsymbol{R}_{\mathrm{ab}}\} = \begin{Bmatrix} X_{ab} \\ Z_{ab} \\ M_{ab} \\ X_{ba} \\ Z_{ba} \\ M_{ba} \end{Bmatrix} = \begin{Bmatrix} -5000 \\ -8660 \\ 12990 \\ -5000 \\ -8660 \\ -12990 \end{Bmatrix} + \\ + \begin{pmatrix} 1.44 & 0 & 0 & -1.44 & 0 & 0 \\ 0 & 0.0144 & -0.0432 & 0 & -0.0144 & -0.0432 \\ 0 & -0.0432 & 0.1728 & 0 & 0.0432 & 0.0864 \\ -1.44 & 0 & 0 & 1.44 & 0 & 0 \\ 0 & -0.0144 & 0.0432 & 0 & 0.0144 & 0.0432 \\ 0 & -0.0432 & 0.0864 & 0 & 0.0432 & 0.1728 \end{pmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1.736 \\ 0 \\ 19.905 \end{Bmatrix} \cdot 10^3$$

$$\{\boldsymbol{R}_{\mathrm{bc}}\} = \begin{Bmatrix} X_{bc} \\ Z_{bc} \\ M_{bc} \\ X_{cb} \\ Z_{cb} \\ M_{cb} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -11296 \\ 6111 \\ 0 \\ 1296 \\ 1666 \end{Bmatrix} + \\ + \begin{pmatrix} 1.44 & 0 & 0 & -1.44 & 0 & 0 \\ 0 & 0.0144 & -0.0432 & 0 & -0.0144 & -0.0432 \\ 0 & -0.0432 & 0.1728 & 0 & 0.0432 & 0.0864 \\ -1.44 & 0 & 0 & 1.44 & 0 & 0 \\ 0 & -0.0144 & 0.0432 & 0 & 0.0144 & 0.0432 \\ 0 & -0.0432 & 0.0864 & 0 & 0.0432 & 0.1728 \end{pmatrix} \begin{Bmatrix} 1.736 \\ 0 \\ 19.905 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \cdot 10^3$$

There is value 10^3 that we obtain by multiplying stiffness matrix multiplier 10^9 and displacement vector multiplier 10^{-6}.

$$\{\boldsymbol{R}_{\mathrm{ab}}\} = \begin{Bmatrix} X_{\mathrm{ab}} \\ Z_{\mathrm{ab}} \\ M_{\mathrm{ab}} \\ X_{\mathrm{ba}} \\ Z_{\mathrm{ba}} \\ M_{\mathrm{ba}} \end{Bmatrix} = \begin{Bmatrix} -7500.16 \\ -9519.90 \\ 14709.80 \\ -2500.16 \\ -7800.00 \\ -9550.00 \end{Bmatrix} [\mathrm{N}, \mathrm{Nm}]$$

$$\{\boldsymbol{R}_{\mathrm{bc}}\} = \begin{Bmatrix} X_{\mathrm{bc}} \\ Z_{\mathrm{bc}} \\ M_{\mathrm{bc}} \\ X_{\mathrm{cb}} \\ Z_{\mathrm{cb}} \\ M_{\mathrm{cb}} \end{Bmatrix} = \begin{Bmatrix} 2500 \\ -12156 \\ 9550 \\ -2500 \\ 2156 \\ 3385 \end{Bmatrix} [\mathrm{N}, \mathrm{Nm}]$$

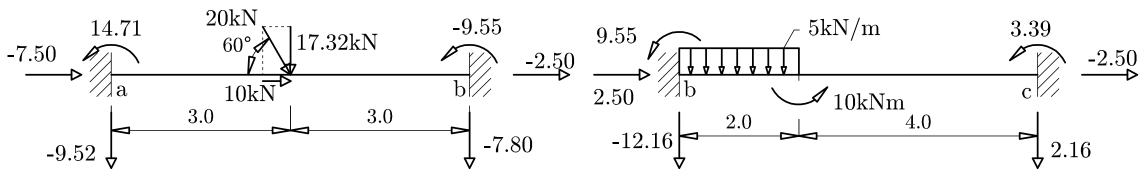

Fig. 4: Force equilibrium on the bar [kN, kNm]

6. Check force equilibrium on the bar – see Fig. 4.

Bar a – b

$$\sum F_x = 0 : -7.50 + 10 - 2.50 = 0$$
$$\sum F_z = 0 : -9.52 + 17.32 - 7.8 = 0$$
$$\sum M_a = 0 : 14.71 - 17.32 \cdot 3 - (-7.8) \cdot 6 + (-9.55) = 0$$

Bar b – c

$$\sum F_x = 0 : 2.50 + (-2.50) = 0$$
$$\sum F_z = 0 : -12.16 + 5 \cdot 2 + 2.16 = 0$$
$$\sum M_b = 0 : 9.55 - \frac{5 \cdot 2^2}{2} + 10 + 3.39 - 2.16 \cdot 6 = 0$$

7. Check force equilibrium in nodes a, b and c – see Fig. 5.

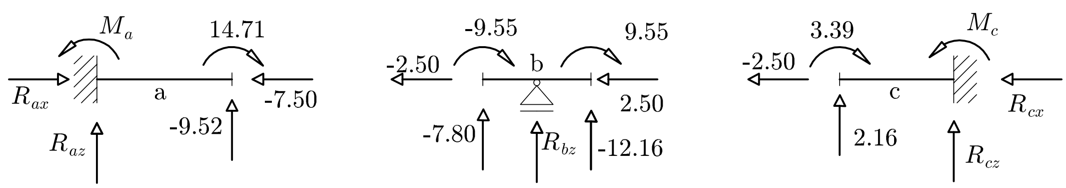

Fig. 5: Force equilibrium in nodes a, b and c [kN, kNm]

Node a

$$\sum F_{xa} = 0 : R_{ax} - (-7.50) = 0 \Rightarrow R_{ax} = -7.50\,\mathrm{kN}$$
$$\sum F_{za} = 0 : R_{az} - (-9.52) = 0 \Rightarrow R_{az} = -9.52\,\mathrm{kN}$$
$$\sum M_a = 0 : M_c - 14.71 = 0 \Rightarrow M_a = 14.71\,\mathrm{kNm}$$

Node b

$$\sum F_{xb} = 0 : -2.50 + 2.50 = 0$$
$$\sum F_{zb} = 0 : R_{bz} - 7.80 - 12.16 = 0 \Rightarrow R_{bz} = 19.96\,\mathrm{kN}$$
$$\sum M_b = 0 : -9.55 + 9.55 = 0$$

Node c

$$\sum F_{xc} = 0 : R_{cx} - (-2.50) = 0 \Rightarrow R_{cx} = -2.50\,\mathrm{kN}$$
$$\sum F_{zc} = 0 : R_{cz} + 2.16 = 0 \Rightarrow R_{cz} = -2.16\,\mathrm{kN}$$
$$\sum M_c = 0 : M_c - 3.39 = 0 \Rightarrow M_c = 3.39\,\mathrm{kNm}$$

8. Internal forces diagrams (N – normal force, V – shear force, M – bending moment) – see Fig. 6.

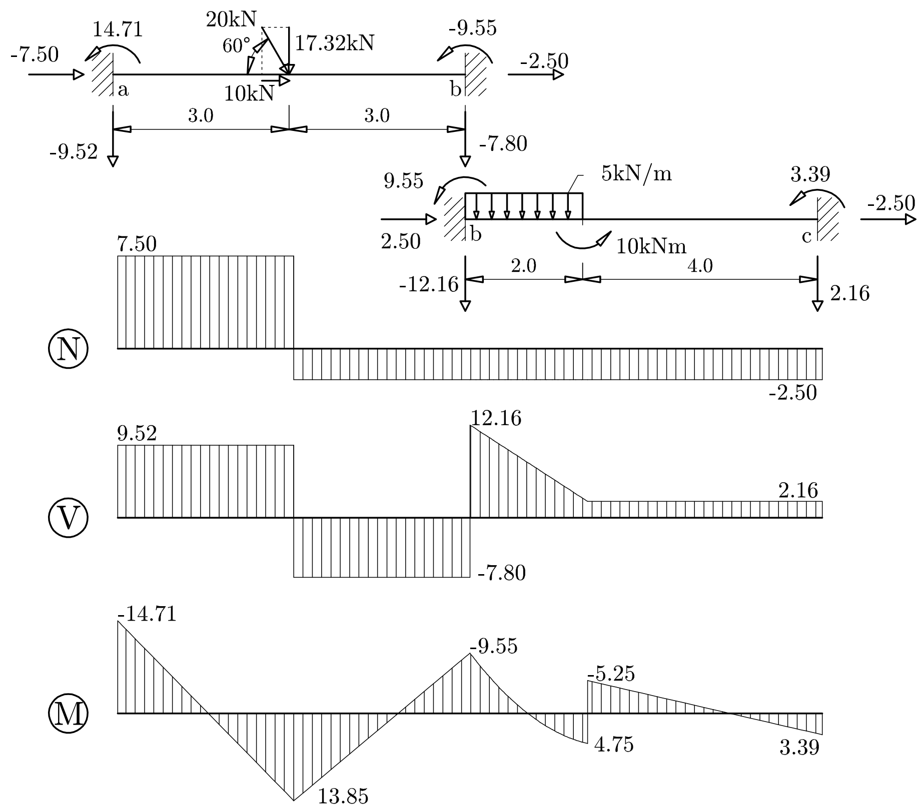

Fig. 6: Internal forces

1.2 2D Frame

Use Direct Stiffness Method to calculate the internal forces of the frame and draw them. See Fig. 7. Material parameters: modulus of elasticity $E = 24\,\mathrm{GPa}$ and coefficient of thermal expansion $\alpha_t = 10^{-5}\,\mathrm{K}^{-1}$.

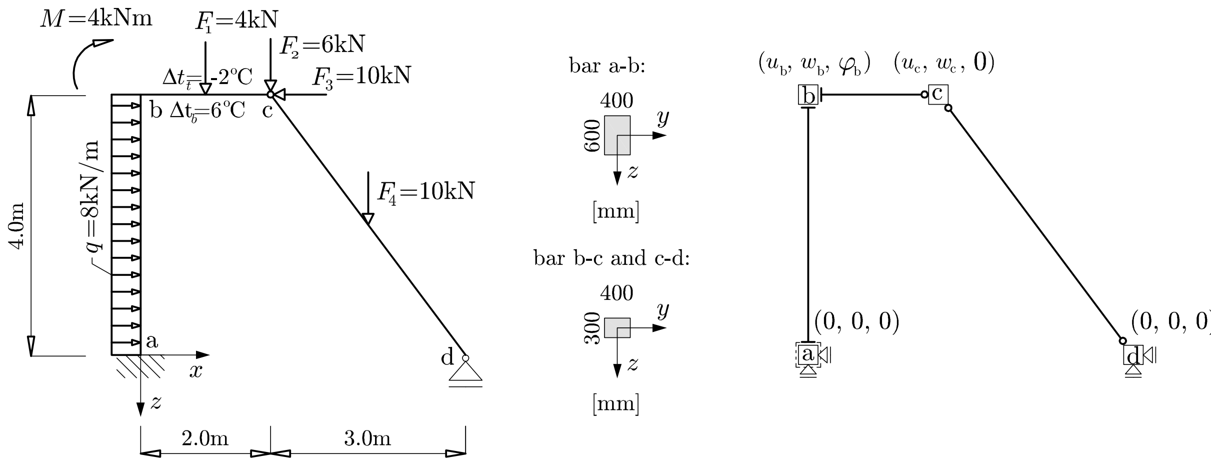

Fig. 7: Left:2D frame, right: model scheme.

Cross-sectional area

$$A_{\mathrm{ab}} = b \cdot h = 0.4 \cdot 0.6 = 0.24\,\mathrm{m}^2$$
$$A_{\mathrm{bc}} = A_{\mathrm{cd}} = b \cdot h = 0.4 \cdot 0.3 = 0.12\,\mathrm{m}^2$$

The second moment of area

$$I_{\mathrm{y}}^{\mathrm{ab}} = \frac{1}{12} \cdot b \cdot h^3 = \frac{1}{12} \cdot 0.4 \cdot 0.6^3 = 0.0072\,\mathrm{m}^4$$
$$I_{\mathrm{y}}^{\mathrm{bc}} = I_{\mathrm{y}}^{\mathrm{cd}} = \frac{1}{12} \cdot b \cdot h^3 = \frac{1}{12} \cdot 0.4 \cdot 0.3^3 = 0.0009\,\mathrm{m}^4$$

1. degree of kinematic indeterminacy (number of degrees of freedom)

$$n_k = 5$$

2. displacement vector and nodal force vector

$$\{\boldsymbol{r}\} = \begin{Bmatrix} u_{\mathrm{b}} \\ w_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \\ u_{\mathrm{c}} \\ w_{\mathrm{c}} \end{Bmatrix} \qquad \{\boldsymbol{S}\} = \begin{Bmatrix} 0 \\ 0 \\ -4\,000 \\ -10\,000 \\ 6\,000 \end{Bmatrix} [\mathrm{N}, \mathrm{Nm}] \tag{9}$$

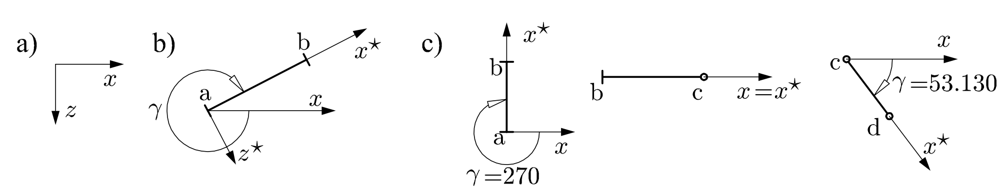

Fig. 8: a) global coordinate system xz, b) local coordinate system of the general bar $x^\star z^\star$ and determination of angle γ (clockwise) c) angle γ of our bars a-b, b-c and c-d

3. stiffness matrix in global coordinate system

(a) bar a-b — tab. 11.4a, angle $\gamma = 270° \Rightarrow s = -1, c = 0$

$$[\boldsymbol{k}_{\mathrm{ab}}] =$$

$$\begin{array}{cccccccc}
u_{\mathrm{a}} & w_{\mathrm{a}} & \varphi_{\mathrm{a}} & u_{\mathrm{b}} & w_{\mathrm{b}} & \varphi_{\mathrm{b}} & & \\
\left(\begin{matrix} . \\ . \\ . \\ . \\ . \\ . \end{matrix}\right. & \begin{matrix} . \\ . \\ . \\ . \\ . \\ . \end{matrix} & \begin{matrix} . \\ . \\ . \\ . \\ . \\ . \end{matrix} & \begin{matrix} -32.4 \\ 0 \\ 64.8 \\ 32.4 \\ 0 \\ 64.8 \end{matrix} & \begin{matrix} 0 \\ -1440 \\ 0 \\ 0 \\ 1440 \\ 0 \end{matrix} & \left.\begin{matrix} -64.8 \\ 0 \\ 86.4 \\ 64.8 \\ 0 \\ 172.8 \end{matrix}\right) & \begin{matrix} u_{\mathrm{a}} \\ w_{\mathrm{a}} \\ \varphi_{\mathrm{a}} \\ u_{\mathrm{b}} \\ w_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \end{matrix} & \left[\cdot 10^6 \frac{\mathrm{N}}{\mathrm{m}}\right]
\end{array}$$

(b) bar b-c – local coordinate system of the bar is same as global coordinate of the system — we can use tab. 11.3b

$$[\boldsymbol{k}_{\mathrm{bc}}] = [\boldsymbol{k}^{\star}_{\mathrm{bc}}] =$$

$$\begin{array}{cccccccc}
u_{\mathrm{b}} & w_{\mathrm{b}} & \varphi_{\mathrm{b}} & u_{\mathrm{c}} & w_{\mathrm{c}} & \varphi_{\mathrm{c}} & & \\
\left(\begin{matrix} 1440 \\ 0 \\ 0 \\ -1440 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 8.1 \\ -16.2 \\ 0 \\ -8.1 \\ 0 \end{matrix} & \begin{matrix} 0 \\ -16.2 \\ 32.4 \\ 0 \\ 16.2 \\ 0 \end{matrix} & \begin{matrix} -1440 \\ 0 \\ 0 \\ 1440 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ -8.1 \\ 16.2 \\ 0 \\ 8.1 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right) & \begin{matrix} u_{\mathrm{b}} \\ w_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \\ u_{\mathrm{c}} \\ w_{\mathrm{c}} \\ \varphi_{\mathrm{c}} \end{matrix} & \left[\cdot 10^6 \frac{\mathrm{N}}{\mathrm{m}}\right]
\end{array}$$

(c) bar c-d — tab. 11.4d, angle $\gamma = 53.130° \Rightarrow s = \frac{4}{5}, c = \frac{3}{5}$

$$[\boldsymbol{k}_{\mathrm{cd}}] =$$

$$\begin{array}{cccccccc}
u_{\mathrm{c}} & w_{\mathrm{c}} & \varphi_{\mathrm{c}} & u_{\mathrm{d}} & w_{\mathrm{d}} & \varphi_{\mathrm{d}} & & \\
\left(\begin{matrix} 207.36 \\ 276.48 \\ 0 \\ -207.36 \\ -276.48 \\ 0 \end{matrix}\right. & \begin{matrix} 276.48 \\ 368.64 \\ 0 \\ -276.48 \\ -368.64 \\ 0 \end{matrix} & \begin{matrix} . \\ . \\ . \\ . \\ . \\ . \end{matrix} & \begin{matrix} . \\ . \\ . \\ . \\ . \\ . \end{matrix} & \begin{matrix} . \\ . \\ . \\ . \\ . \\ . \end{matrix} & \left.\begin{matrix} . \\ . \\ . \\ . \\ . \\ . \end{matrix}\right) & \begin{matrix} u_{\mathrm{c}} \\ w_{\mathrm{c}} \\ \varphi_{\mathrm{c}} \\ u_{\mathrm{d}} \\ w_{\mathrm{d}} \\ \varphi_{\mathrm{d}} \end{matrix} & \left[\cdot 10^6 \frac{\mathrm{N}}{\mathrm{m}}\right]
\end{array}$$

(d) stiffness matrix of the system

$$[\boldsymbol{k}] =$$

$$\begin{array}{ccccccc}
u_{\mathrm{b}} & w_{\mathrm{b}} & \varphi_{\mathrm{b}} & u_{\mathrm{c}} & w_{\mathrm{c}} & & \\
\left(\begin{matrix} 32.4 + 1440 \\ 0 + 0 \\ 64.8 + 0 \\ -1440 \\ 0 \end{matrix}\right. & \begin{matrix} 0 + 0 \\ 1440 + 8.1 \\ 0 + (-16.2) \\ 0 \\ -8.1 \end{matrix} & \begin{matrix} 64.8 + 0 \\ 0 + (-16.2) \\ 172.8 + 32.4 \\ 0 \\ 16.2 \end{matrix} & \begin{matrix} -1440 \\ 0 \\ 0 \\ 1440 + 207.36 \\ 0 + 276.48 \end{matrix} & \left.\begin{matrix} 0 \\ -8.1 \\ 16.2 \\ 0 + 276.48 \\ 8.1 + 368.64 \end{matrix}\right) & \begin{matrix} u_{\mathrm{b}} \\ w_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \\ u_{\mathrm{c}} \\ w_{\mathrm{c}} \end{matrix} & \left[\cdot 10^6 \frac{\mathrm{N}}{\mathrm{m}}\right]
\end{array}$$

$$[\boldsymbol{k}] =$$

$$\begin{array}{cccccccc} & u_\mathrm{b} & w_\mathrm{b} & \varphi_\mathrm{b} & u_\mathrm{c} & w_\mathrm{c} & & \\ & \left(\begin{array}{ccccc} 1472.4 & 0 & 64.8 & -1440 & 0 \\ 0 & 1448.1 & -16.2 & 0 & -8.1 \\ 64.8 & -16.2 & 205.2 & 0 & 16.2 \\ -1440 & 0 & 0 & 1647.36 & 276.48 \\ 0 & -8.1 & 16.2 & 276.48 & 376.74 \end{array}\right. & & & & \left.\right) & \begin{array}{c} u_\mathrm{b} \\ w_\mathrm{b} \\ \varphi_\mathrm{b} \\ u_\mathrm{c} \\ w_\mathrm{c} \end{array} & \left[\cdot 10^6\,\frac{\mathrm{N}}{\mathrm{m}}\right] \end{array}$$

4. primary vectors of end reactions in local coordinates of the bar
   (a) bar a-b — tab 11.2a (tab. 14.10 row 13)

$$\left\{\bar{\boldsymbol{R}}^\star_{ab}\right\} = \left\{\begin{array}{c} \bar{X}^\star_\mathrm{ab} \\ \bar{Z}^\star_\mathrm{ab} \\ \bar{M}^\star_\mathrm{ab} \\ \bar{X}^\star_\mathrm{ba} \\ \bar{Z}^\star_\mathrm{ba} \\ \bar{M}^\star_\mathrm{ba} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ -16 \\ 10.\bar{6} \\ 0 \\ -16 \\ -10.\bar{6} \end{array}\right\} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{10}$$

– transformation to the global coordinate system

$$\left\{\bar{\boldsymbol{R}}_{ab}\right\} = \left\{\begin{array}{c} \bar{X}_\mathrm{ab} \\ \bar{Z}_\mathrm{ab} \\ \bar{M}_\mathrm{ab} \\ \bar{X}_\mathrm{ba} \\ \bar{Z}_\mathrm{ba} \\ \bar{M}_\mathrm{ba} \end{array}\right\} = \left\{\begin{array}{c} -16 \\ 0 \\ 10.\bar{6} \\ -16 \\ 0 \\ -10.\bar{6} \end{array}\right\} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{11}$$

   (b) bar b-c — force F_1: tab. 11.2c (tab. 14.11 row 2) and thermal load t: tab. 11.5b

$$\left\{\bar{\boldsymbol{R}}^\star_{bc}\right\}_{\mathrm{F}_1} = \left\{\bar{\boldsymbol{R}}_{bc}\right\}_{\mathrm{F}_1} = \left\{\begin{array}{c} 0 \\ -2.75 \\ 1.5 \\ 0 \\ -1.25 \\ 0 \end{array}\right\} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{12}$$

$$\left\{\bar{\boldsymbol{R}}^\star_{bc}\right\}_{\mathrm{t}} = \left\{\bar{\boldsymbol{R}}_{bc}\right\}_{\mathrm{t}} = \left\{\begin{array}{c} 57.6 \\ -4.32 \\ 8.64 \\ -57.6 \\ 4.32 \\ 0 \end{array}\right\} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{13}$$

$$\left\{\bar{\boldsymbol{R}}^\star_{bc}\right\} = \left\{\bar{\boldsymbol{R}}_{bc}\right\} = \left\{\begin{array}{c} \bar{X}^\star_\mathrm{bc} \\ \bar{Z}^\star_\mathrm{bc} \\ \bar{M}^\star_\mathrm{bc} \\ \bar{X}^\star_\mathrm{cb} \\ \bar{Z}^\star_\mathrm{cb} \\ \bar{M}^\star_\mathrm{cb} \end{array}\right\} = \left\{\begin{array}{c} 57.6 \\ -7.07 \\ 10.14 \\ -57.6 \\ 3.07 \\ 0 \end{array}\right\} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{14}$$

(c) bar c-d — similar to simply supported beam

$$\left\{\bar{\boldsymbol{R}}^{\star}_{cd}\right\} = \begin{Bmatrix} \bar{X}^{\star}_{\mathrm{cd}} \\ \bar{Z}^{\star}_{\mathrm{cd}} \\ \bar{M}^{\star}_{\mathrm{cd}} \\ \bar{X}^{\star}_{\mathrm{dc}} \\ \bar{Z}^{\star}_{\mathrm{dc}} \\ \bar{M}^{\star}_{\mathrm{dc}} \end{Bmatrix} = \begin{Bmatrix} -4 \\ -3 \\ 0 \\ -4 \\ -3 \\ 0 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{15}$$

– transformation to the global coordinate system

$$\left\{\bar{\boldsymbol{R}}_{cd}\right\} = \begin{Bmatrix} \bar{X}_{\mathrm{cd}} \\ \bar{Z}_{\mathrm{cd}} \\ \bar{M}_{\mathrm{cd}} \\ \bar{X}_{\mathrm{dc}} \\ \bar{Z}_{\mathrm{dc}} \\ \bar{M}_{\mathrm{dc}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -5 \\ 0 \\ 0 \\ -5 \\ 0 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{16}$$

(d) primary vector of the system

$$\left\{\bar{\boldsymbol{R}}\right\} = \begin{Bmatrix} -16 + 57.6 \\ 0 + (-7.07) \\ -10.\bar{6} + 10.14 \\ -57.6 + 0 \\ 3.07 + (-5) \end{Bmatrix} \cdot 10^3 = \begin{Bmatrix} 41.6 \\ -7.07 \\ -0.52\bar{6} \\ -57.6 \\ -1.93 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{17}$$

5. force vector

$$\{\boldsymbol{F}\} = \{\boldsymbol{S}\} - \left\{\bar{\boldsymbol{R}}\right\} = \begin{Bmatrix} 0 \\ 0 \\ -4 \\ -10 \\ 6 \end{Bmatrix} \cdot 10^3 - \begin{Bmatrix} 41.6 \\ -7.07 \\ -0.52\bar{6} \\ -57.6 \\ -1.93 \end{Bmatrix} \cdot 10^3 = \begin{Bmatrix} -41.6 \\ 7.07 \\ -3.47\bar{3} \\ 47.6 \\ 7.93 \end{Bmatrix} \left[\cdot 10^3\,\mathrm{N}, \mathrm{Nm}\right] \tag{18}$$

6. equation system

$$[\boldsymbol{k}] \cdot \{\boldsymbol{r}\} = \{\boldsymbol{F}\} \tag{19}$$

$$\begin{pmatrix} 1472.4 & 0 & 64.8 & -1440 & 0 \\ 0 & 1448.1 & -16.2 & 0 & -8.1 \\ 64.8 & -16.2 & 205.2 & 0 & 16.2 \\ -1440 & 0 & 0 & 1647.36 & 276.48 \\ 0 & -8.1 & 16.2 & 276.48 & 376.74 \end{pmatrix} \cdot 10^6 \cdot \begin{Bmatrix} u_{\mathrm{b}} \\ w_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \\ u_{\mathrm{c}} \\ w_{\mathrm{c}} \end{Bmatrix} = \begin{Bmatrix} -41.6 \\ 7.07 \\ -3.47\bar{3} \\ 47.6 \\ 7.93 \end{Bmatrix} \cdot 10^3$$

$$\begin{Bmatrix} u_{\mathrm{b}} \\ w_{\mathrm{b}} \\ \varphi_{\mathrm{b}} \\ u_{\mathrm{c}} \\ w_{\mathrm{c}} \end{Bmatrix} = \begin{Bmatrix} 39.171 \\ 4.429 \\ -26.783 \\ 67.736 \\ -27.414 \end{Bmatrix} \left[\cdot 10^{-6}\,\mathrm{m}, \mathrm{rad}\right]$$

7. End reaction vector of the bar

$$\{\boldsymbol{R}_{\mathrm{ab}}\} = \left\{\bar{\boldsymbol{R}}_{\mathrm{ab}}\right\} + \left\{\hat{\boldsymbol{R}}_{\mathrm{ab}}\right\} = \left\{\bar{\boldsymbol{R}}_{\mathrm{ab}}\right\} + [\boldsymbol{k}_{\mathrm{ab}}]\,\{\boldsymbol{r}_{\mathrm{ab}}\} \tag{20}$$

$$\left\{\hat{\boldsymbol{R}}_{\mathrm{ab}}\right\} = \begin{pmatrix} . & . & . & -32.4 & 0 & -64.8 \\ . & . & . & 0 & -1440 & 0 \\ . & . & . & 64.8 & 0 & 86.4 \\ . & . & . & 32.4 & 0 & 64.8 \\ . & . & . & 0 & 1440 & 0 \\ . & . & . & 64.8 & 0 & 172.8 \end{pmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 39.171 \\ 4.429 \\ -26.783 \end{Bmatrix} = \begin{Bmatrix} 466.398 \\ -6377.76 \\ 224.230 \\ -466.398 \\ 6377.76 \\ -2089.822 \end{Bmatrix}$$

$$\left\{\hat{\boldsymbol{R}}_{\mathrm{bc}}\right\} = \begin{pmatrix} 1440 & 0 & 0 & -1440 & 0 & 0 \\ 0 & 8.1 & -16.2 & 0 & -8.1 & 0 \\ 0 & -16.2 & 32.4 & 0 & 16.2 & 0 \\ -1440 & 0 & 0 & 1440 & 0 & 0 \\ 0 & -8.1 & 16.2 & 0 & 8.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{Bmatrix} 39.171 \\ 4.429 \\ -26.783 \\ 67.736 \\ -27.414 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -41133.6 \\ 691.813 \\ -1383.626 \\ 41133.6 \\ -691.813 \\ 0 \end{Bmatrix}$$

$$\left\{\hat{\boldsymbol{R}}_{\mathrm{cd}}\right\} = \begin{pmatrix} 207.36 & 276.48 & . & . & . & . \\ 276.48 & 368.64 & . & . & . & . \\ 0 & 0 & . & . & . & . \\ -207.36 & -276.48 & . & . & . & . \\ -276.48 & -368.64 & . & . & . & . \\ 0 & 0 & . & . & . & . \end{pmatrix} \cdot \begin{Bmatrix} 67.736 \\ -27.414 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6466.314 \\ 8621.752 \\ 0 \\ -6466.314 \\ -8621.752 \\ 0 \end{Bmatrix}$$

$$\{\boldsymbol{R}_{\mathrm{ab}}\} = \begin{Bmatrix} -15533.602 \\ -6377.76 \\ 10890.89\bar{6} \\ -16466.398 \\ 6377.76 \\ -12765.488\bar{6} \end{Bmatrix} \quad \xrightarrow[\text{transform}]{} \quad \{\boldsymbol{R}^{\star}_{\mathrm{ab}}\} = \begin{Bmatrix} 6377.76 \\ -15533.6 \\ 10890.89\bar{6} \\ -6377.76 \\ -16466.398 \\ -12765.488\bar{6} \end{Bmatrix}$$

$$\{\boldsymbol{R}_{\mathrm{bc}}\} = \{\boldsymbol{R}^{\star}_{\mathrm{bc}}\} = \begin{Bmatrix} 16466.4 \\ -6378.187 \\ 8756.374 \\ -16466.4 \\ 2378.187 \\ 0 \end{Bmatrix}$$

$$\{\boldsymbol{R}_{\mathrm{cd}}\} = \begin{Bmatrix} 6466.314 \\ 3621.752 \\ 0 \\ -6466.314 \\ -13621.752 \\ 0 \end{Bmatrix} \quad \xrightarrow[\text{transform}]{} \quad \{\boldsymbol{R}^{\star}_{\mathrm{cd}}\} = \begin{Bmatrix} 6777.19 \\ -3000 \\ 0 \\ -14777.19 \\ -3000 \\ 0 \end{Bmatrix}$$

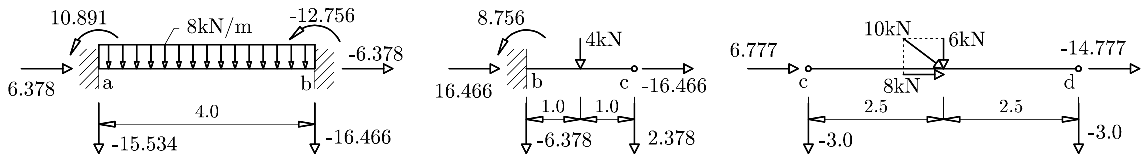

Fig. 9: Force equilibrium on the bar [kN, kNm]

8. Check force equilibrium on the bar – see Fig. 4.

Bar a – b

$$\sum F_{\mathrm{x}} = 0 : 6.378 + (-6.378) = 0$$
$$\sum F_{\mathrm{z}} = 0 : -15.534 + 8 \cdot 4 - 16.466 = 0$$
$$\sum M_{\mathrm{a}} = 0 : 10.891 - 8 \cdot \frac{4^2}{2} + (-12.756) - (-16.466) \cdot 4 = 0$$

Bar b – c

$$\sum F_{\mathrm{x}} = 0 : 16.466 + (-16.466) = 0$$
$$\sum F_{\mathrm{z}} = 0 : -6.378 + 4 + 2.378 = 0$$
$$\sum M_{\mathrm{b}} = 0 : 8.756 - 4 \cdot 1 - 2.378 \cdot 2 = 0$$

Bar c – d

$$\sum F_{\mathrm{x}} = 0 : 6.777 + 8 + (-14.777) = 0$$
$$\sum F_{\mathrm{z}} = 0 : -3 + 6 + (-3) = 0$$
$$\sum M_{\mathrm{b}} = 0 : 6 \cdot 2.5 + (-3) \cdot 5 = 0$$

9. Check force equilibrium in nodes a, b and c and support reactions – see Fig. 5.

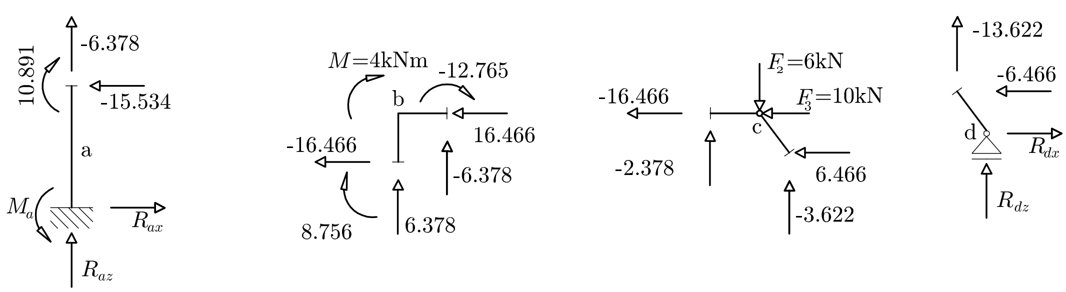

Fig. 10: Force equilibrium in nodes a, b, c and d [kN, kNm]

Node a

$$\sum F_{xa} = 0 : R_{\mathrm{ax}} - 6.378 = 0 \Rightarrow R_{\mathrm{ax}} = 6.378\,\mathrm{kN}$$
$$\sum F_{za} = 0 : R_{\mathrm{az}} + (-15.534) = 0 \Rightarrow R_{\mathrm{az}} = 15.534\,\mathrm{kN}$$
$$\sum M_{\mathrm{a}} = 0 : M_a - 10.891 = 0 \Rightarrow M_{\mathrm{a}} = 10.891\,\mathrm{kNm}$$

Node b

$$\sum F_{\mathrm{xb}} = 0 : -16.466 + 16.466 = 0$$
$$\sum F_{\mathrm{zb}} = 0 : 6.378 + (-6.378) = 0$$
$$\sum M_{\mathrm{b}} = 0 : 8.756 + 4 + (-12.765) = 0$$

Node c

$$\sum F_{xc} = 0 : -16.466 + 10 + 6.466 = 0$$

$$\sum F_{\mathrm{zc}} = 0 : -2.378 + 6 + (-3.622) = 0$$

Node d

$$\sum F_{\mathrm{xd}} = 0 : R_{\mathrm{dx}} - (-6.466) = 0 \Rightarrow R_{\mathrm{dx}} = -6.466\,\mathrm{kN}$$

$$\sum F_{\mathrm{zd}} = 0 : R_{\mathrm{dz}} + (-13.622) = 0 \Rightarrow R_{\mathrm{dz}} = -13.622\,\mathrm{kN}$$

10. Internal forces diagrams (N – normal force, V – shear force, M – bending moment) – see Fig. 6.

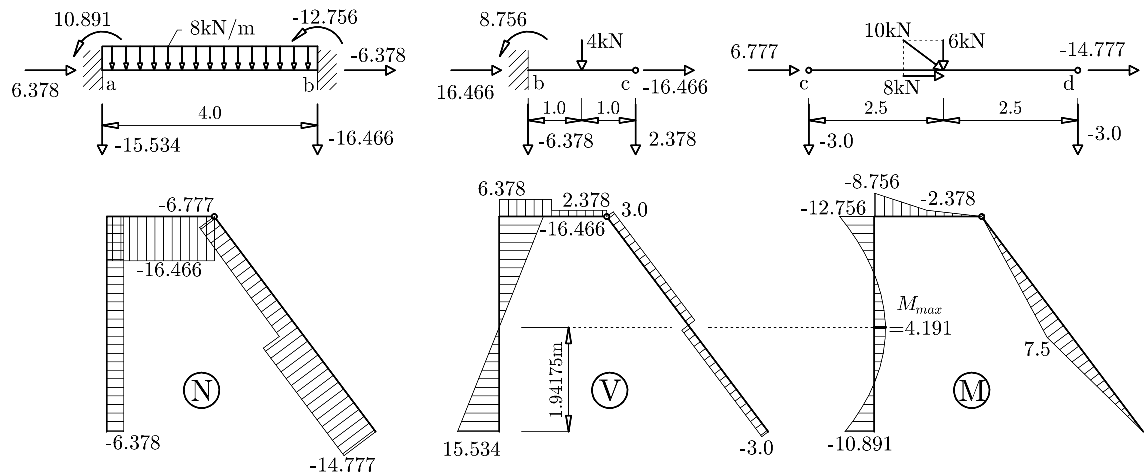

Fig. 11: Internal forces of the frame – N, V, M